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VORTEX REGIONS IN A POTENTIAL STREAM WITH A JUMP OF BERNOULLI'S CONSTANT AT THE BOUNDARY

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The problem of "splicing" of a vortex flow in a certain finite region of an incompressible fluid with the surrounding potential stream along a fluid streamline is considered in the case in which the Bernoulli constant is subject to discontinuity of a given magnitude along the streamline separating these two flows. A solution is found in the form of integrals containing two unknown functions for the definition of the contour and the vortex sheet intensity. A system of two nonlinear integral equations is derived for the determination of these parameters and the results of certain computer calculations are presented.

Some of the recent models of incompressible fluid flow with zones of separation at high Reynolds numbers [1, 2] show that the limit solution of the Navier-Stokes equations defines a flow with a constant vortex in the separation zone (in the case of plane flow) bordering on the external potential stream. This has prompted a number of investigations of vortex and potential flows in contact along a fluid streamline. The problem of such flow in a given finite region is considered in [3]. A similar problem of flow in an unbounded region is considered in [4 - 6], and an application of this solution to the investigation of flows past bodies with stationary separation zones at high Reynolds numbers is presented in [7]. The problem of "splicing" of vortex and potential flows in the presence of a body when the Bernoulli constant becomes discontinuous at the vortex zone boundary is examined in [8] in an approximate manner.

Below we present a solution of the exactly formulated problem of "splicing" in the presence of a jump of Bernoulli's constant in a flow without rigid boundaries, which according to [7] corresponds to infinitely great Reynolds numbers and special boundary conditions in the separation zone.

1. Let us consider a two-dimensional stationary potential flow of a perfect incompressible fluid containing a zone Σ of vortex flow. Let the direction of the X-axis coincide with that of the potential stream at infinity and the length of the zone along this axis be equal to l. We specify the vortex distribution by

$$\Omega(x, y) = -\omega_0 \operatorname{sign} y \qquad (\omega_0 = \text{const} > 0)$$

and introduce in the usual manner the stream function ψ

$$\partial\psi/\partial x = -v, \qquad \partial\psi/\partial y = u$$

where u and v are velocity components along the X - and the Y -axes, respectively, normalized as all other magnitudes with respect to ω_0 and $^1/_2\, l$. This stream function satisfies the Laplace equation and the Poisson equation $\nabla^2\psi = \operatorname{sign} y$, respectively, outside region Σ and inside it. Let the Bernoulli constant be discontinuous at the boundary streamline L (boundary of region Σ).

By using the pressure continuity condition and the Bernoulli equation it can be shown that along L, $V_e^2 - V_i^2 = h$ where V is the velocity and h a constant equal to double the jump of Bernoulli's constant at the boundary streamline. Subscripts i and e relate to the vortex and the potential flows respectively. Since velocity is equal to within the sign to the derivative of ψ along a normal to the streamline, hence

$$h = (\partial\psi/\partial n_e)^2 - (\partial\psi/\partial n_i)^2 \tag{1.1}$$

where the right-hand side contains the limit values of the normal derivative at the boundary of region Σ and directed toward the potential flow.

Let us consider the following problem. We have to determine the stream function $\psi(x, y)$ harmonic outside region Σ and satisfying equation $\nabla^2\psi = \operatorname{sign} y$ inside it for boundary conditions as follows: (a) along the contour L, which is yet to be determined, we must have $\psi_e = \psi_i = \psi(x, 0) = \text{const}$ and the condition (1.1) must be satisfied for specified values of h; (b) at infinity $\partial\psi/\partial x = 0$ and $\partial\psi/\partial y = V_\infty$ with the latter also remaining to be determined.

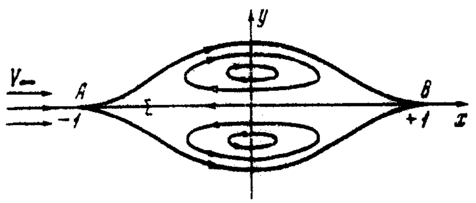

Fig. 1.

2. Let Σ be the unknown region and let the curvature of its boundary satisfy the Hölder boundary conditions everywhere, except in the small region containing points A and B (Fig. 1). We seek function $\psi(x, y)$ in the form

$$\psi = \psi_0 + \psi_1 + \psi_2 \tag{2.1}$$

where ψ_0 is the stream function of a uniform flow at velocity V_∞ and ψ_1 is the stream function of the flow induced by vortices emanating from region Σ. The purpose of the third term in (2.1) is to produce a discontinuity of velocity along the boundary streamline and represents the logarithmic potential of a simple layer (from the point of view of hydrodynamics this term corresponds to the stream function of the vortex sheet spread along boundary L). We denote the intensity of the vortex sheet by $\Gamma(x)$. The terms in (2.1) can now be written as

$$\psi_0 = V_\infty y, \quad \psi_1 = \frac{1}{2\pi}\iint\limits_{\Sigma} \operatorname{sign}\eta \ln r \, d\xi \, d\eta$$
$$\psi_2 = \frac{1}{2\pi}\oint\limits_{L} \Gamma(\xi)\operatorname{sign}\eta \ln r \, dl, \quad r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \tag{2.2}$$

where $\psi_1(x, y)$ is the logarithmic surface potential satisfying the equation of Laplace and $\nabla^2\psi_1 = \operatorname{sign} y$ outside and inside of region Σ respectively. Its first derivatives (i.e. the velocity vector components) are continuous everywhere including the boundary. Similarly $\psi_2(x, y)$ is a function continuous at the boundary L and harmonic everywhere except at points of the contour. The limit values of its normal derivative satisfy the relationships

$$\frac{\partial\psi_2}{\partial n_e}-\frac{\partial\psi_2}{\partial n_i}=\Gamma\operatorname{sign}y,\qquad \frac{\partial\psi_2}{\partial n_e}+\frac{\partial\psi_2}{\partial n_i}=2\frac{\partial\psi_2}{\partial n_0} \tag{2.3}$$

$$\frac{\partial\psi_2}{\partial n_0}\bigg|_K=-\frac{1}{2\pi}\oint\limits_L\frac{\Gamma(\xi)\cos\varphi_0\operatorname{sign}\eta\,dl}{r} \tag{2.4}$$

where φ_0 is the angle between the normal to the boundary at point K and the position vecter r from that point to the point of integration. Using Eqs. (2.1) and (2.3) for the discontinuity of velocity at the boundary streamline, we obtain

$$V_e-V_i=[V]=\left[\frac{\partial\psi}{\partial n}\right]\operatorname{sign}y=\left[\frac{\partial\psi_2}{\partial n}\right]\operatorname{sign}y=\Gamma(x) \tag{2.5}$$

Since the first derivatives of the logarithmic potential of the surface and of the simple layer vanish at infinity, hence $\partial\psi/\partial x=0$ and $\partial\psi/\partial y=V_\infty$ when $(x, y)=\infty$. Thus (2.1) and (2.2) provide the solution of our problem, provided that contour L and the vortex sheet intensity $\Gamma(x)$ necessary for satisfying condition (1.1) can be determined.

Let region Σ be symmetric about the X-axis. From (2.1) and (2.2) we then readily find that $\psi(x, 0)\equiv 0$, and consequently the boundary streamline $y=f(x)$ must satisfy equation

$$yV_\infty+\psi_1(x, y)+\psi_2(x, y)=0$$

A single integration of $\psi_1(x, y)$ reduces this integral equation to

$$y=-\frac{1}{4\pi V_\infty}\int\limits_{-1}^{1}\Big\{2(\xi-x)\left[\operatorname{arctg}\frac{\eta-y}{\xi-x}-\operatorname{arctg}\frac{\eta+y}{\xi-x}\right]-$$

$$-(\eta+y+\Gamma\sqrt{1+\eta'^2})\ln[(\xi-x)^2+(\eta+y)^2]\Big\}d\xi-$$

$$-\frac{1}{4\pi V_\infty}\left\{\int\limits_{-1}^{1}(\eta-y+\Gamma\sqrt{1+\eta'^2})\ln[(\xi-x)^2+(\eta-y)^2]\,d\xi-I\right\} \tag{2.6}$$

$$I=2\{2y-y\{(1+x)\ln[(1+x)^2+y^2]+(1-x)\ln[(1-x)^2+y^2]\}-$$

$$-y^2\left[\operatorname{arctg}\frac{1+x}{y}+\operatorname{arctg}\frac{1-x}{y}\right]-(1+x)^2\operatorname{arctg}\frac{y}{1+x}-$$

$$-(1-x)^2\operatorname{arctg}\frac{y}{1-x}\Big\},\qquad \eta'=\frac{df(\xi)}{d\xi}$$

where (ξ, η) and (x, y) are points of contour L and the expression for V_∞ in terms of $\Gamma(x)$ and $f(x)$ will be derived later.

We transform condition (1.1) at the region boundary as follows. From (1.1) and (2.5) we have

$$\Gamma(x)=\frac{h}{V_i+V_e}=h\left(\frac{\partial\psi}{\partial n_i}+\frac{\partial\psi}{\partial n_e}\right)^{-1}\operatorname{sign}y$$

Furthermore

$$\frac{\partial\psi}{\partial n_i} + \frac{\partial\psi}{\partial n_e} = 2\frac{\partial\psi_0}{\partial n} + 2\frac{\partial\psi_1}{\partial n} + \frac{\partial\psi_2}{\partial n_e} + \frac{\partial\psi_2}{\partial n_i} = 2\left(\frac{\partial\psi_0}{\partial n} + \frac{\partial\psi_1}{\partial n} + \frac{\partial\psi_2}{\partial n_0}\right)$$

Calculating $\partial\psi_0/\partial n$ and $\partial\psi_1/\partial n$ and using (2.3) and (2.4), we finally obtain

$$\Gamma(x) = \frac{1}{2}h\left|V_\infty\cos\alpha - \frac{1}{2\pi}\oint_L \operatorname{sign}\eta\ln r\cos(n, n_0)\,dl + \right.$$

$$\left. + \frac{\cos\alpha}{2\pi}\int_{-1}^{1}\ln[(\xi - x)^2 + y^2]\,d\xi - \frac{1}{2\pi}\oint_L \frac{\Gamma\cos\varphi_0\operatorname{sign}\eta\,dl}{r}\right|^{-1} \qquad (2.7)$$

where α is the angle of inclination of the tangent to the contour at point (x, y), and n and n_0 are normals to the latter at points (ξ, η) and (x, y) respectively.

Relationships (2.6) and (2.7) represent a system of two nonlinear integral equations for the determination of the unknown $f(x)$ and $\Gamma(x)$

3. Let us examine the boundary conditions which have to be satisfied by $f(x)$ and $\Gamma(x)$. For $h > 0$ (the only case considered here) A and B (Fig. 1) are critical points of the internal flow at which the slope of the boundary streamline must be zero. (Otherwise these points would also be critical in the external flow, which owing to the continuity of pressure and the difference of limit values of Bernoulli's constant along the boundary streamline, is not possible). For $f(x)$ we thus have

$$f(-1) = f(1) = f'(-1) = f'(1) = 0 \qquad (f' = df/dx) \qquad (3.1)$$

Since

$$V_e^2 - V_i^2 = h, \qquad V_e - V_i = \Gamma$$

at the critical points A and B we similarly have

$$\Gamma(-1) = \Gamma(1) = \sqrt{h} \qquad (3.2)$$

Finally, the expression for V_∞ in terms of $f(x)$ and $\Gamma(x)$ can be derived as follows. As shown in [7], it is evident from the relationship $V_e^2 - V_i^2 = h$ that at the critical points $V_e(-1, 0) = V_e(1, 0) = \sqrt{h}$ Furthermore, for any point M lying on the X-axis to the left of point A

$$u(M) = V_\infty + \partial\psi_1/\partial y\,|_M + \partial\psi_2/\partial y\,|_M$$

At the limit with point M approaching point A along the X-axis we have $u(M) \to \to V_e(-1, 0) = \sqrt{h}$ and for V_∞ we obtain

$$V_\infty = \sqrt{h} - \partial\psi_1/\partial y - \partial\psi_2/\partial y \qquad \text{for } x = -1 - 0, \quad y = 0 \qquad (3.3)$$

The calculation of derivative $\partial\psi_1/\partial y$ owing to its continuity, does not present any difficulty

$$\left.\frac{\partial\psi_1}{\partial y}\right|_A = -\frac{1}{2\pi}\int_{-1}^{1}\ln\left[1 + \left(\frac{\eta}{\xi + 1}\right)^2\right]d\xi \qquad (3.4)$$

Passing to the determination of $\partial\psi_2/\partial y$ for $x = -1 - 0$, $y = 0$ we note that

$$\left.\frac{\partial\psi_2}{\partial y}\right|_M = \frac{1}{\pi}\int_{L_1} \frac{\Gamma(\xi)\cos(R, y)\,dl}{R} \equiv I(\delta) \tag{3.5}$$

where L_1 is the upper part of contour L and the remaining notation is made clear in Fig. 2. The following relationship is valid

$$\lim_{\delta\to 0} I(\delta) = I(0) \equiv \frac{1}{\pi}\int_{L_1} \frac{\Gamma(\xi)\cos(r, y)\,dl}{r} \tag{3.6}$$

To prove the validity of (3.6) we make an estimate of the difference

$$|I(\delta) - I(0)| \leqslant \int_{\varepsilon} \Gamma \left| \frac{\cos(R, y)}{R} - \frac{\cos(r, y)}{r}\right| dl + \int_{L_1-\varepsilon} \Gamma \left| \frac{\cos(R, y)}{R} - \frac{\cos(r, y)}{r}\right| dl \tag{3.7}$$

where ε is a fairly small but definite part of contour L_1 comprising point A. The integral in (3.7) along curve $L_1 - \varepsilon$ vanishes for $\delta \to 0$ since its integrand is uniformly continuous along $L_1 - \varepsilon$ and it is possible to pass to the limit in it. Passing to the estimate of the remaining integral, we note that

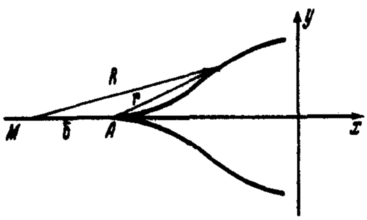

Fig. 2.

$$\lim\,[r^{-1}\cos(r, y)] = {}^1/_2 K_0 \quad \text{for} \quad r \to 0$$

where K_0 is the curvature of contour L_1 at point A [9]. Allowing for this, we derive the following estimates:

$$\int_{\varepsilon} \Gamma(\xi) \left| \frac{\cos(R, y)}{R} - \frac{\cos(r, y)}{r}\right| dl = \int_{\varepsilon} \Gamma \frac{\cos(r, y)}{r}\,\frac{R^2 - r^2}{R^2}\,dl \leqslant$$

$$\leqslant M_0 \int_{\varepsilon} \frac{R^2 - r^2}{R^2}\,dl \leqslant M_0\delta \int_{\varepsilon} \frac{R + r}{R^2}\,dl \leqslant 2M_0\delta \int_0^{r_\varepsilon} \frac{dr}{\sqrt{\delta^2 + r^2}}$$

$$M_0 = \max\left[\frac{\cos(r, y)}{r}\right] \quad \text{при} \ (\xi, \eta) \in \varepsilon$$

Having calculated the definite integral (for $\delta \to 0$) we finally obtain

$$\int_{\varepsilon} \Gamma \left| \frac{\cos(R, y)}{R} - \frac{\cos(r, y)}{r}\right| dl \leqslant 2M_0\delta \ln \frac{r_\varepsilon + \sqrt{r_\varepsilon^2 + \delta^2}}{\delta} \to 0$$

The relationship (3.6) is thus proved.

After transformation, for V_∞ we obtain

$$V_\infty = \sqrt{h_\cdot} + \frac{1}{2\pi}\int_{-1}^{1} \left\{ \ln\left[1 + \left(\frac{\eta}{\xi + 1}\right)^2\right] + \frac{2\Gamma\eta\sqrt{1 + \eta'^2}}{(\xi + 1)^2 + \eta^2}\right\} d\xi \tag{3.8}$$

Note that in the integrand

$$\frac{\eta}{\xi + 1} \to 0, \qquad \frac{\eta}{(\xi + 1)^2 + \eta^2} \to \frac{1}{2} K_0 \quad \text{for} \quad \xi \to -1 + 0$$

Taking into consideration (3.8), we find that the system of Eqs. (2.6) - (2.7) contains only two unknown functions: $f(x)$ and $\Gamma(x)$ which satisfy boundary conditions (3.1) and (3.2).

An analytical proof of the existence and uniqueness (or otherwise) of the solution of this system does not seem possible. Because of this the system was solved numerically on a computer by the method of successive approximations (this explains the reason for the very special form of Eqs. (2.6) and (2.7)). It should be noted that the computer time needed for the calculation of a flow at one particular value of h is very considerable. For this reason only a few data are presented here.

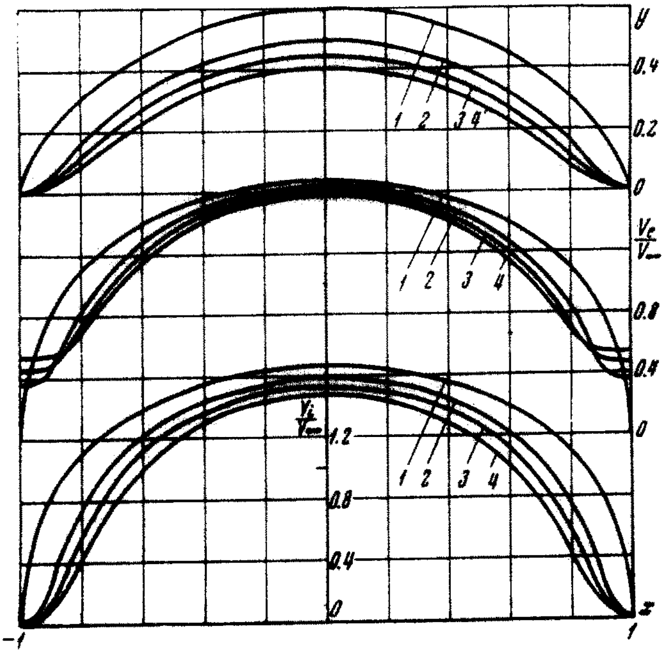

Fig. 3.

Streamlines $\psi = 0$ for the upper half-plane, and the distribution of the vortex sheet $\Gamma(x)$ and of velocity along the contour, calculated for several values of h, are shown in Figs. 3 and 4. Curves 1 - 4 relate to $10^3 h = 0, 2.5, 3.6$ and 4.4, respectively.

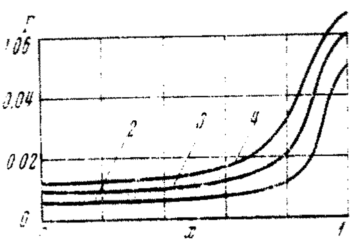

Fig. 4.

The magnitude of vortex ω_0 relative to $2u_\infty/l$, where u_∞ is the dimensional value of velocity at infinity, was found to be for these values of h respectively, 7.063, 7.400, 7.841 and 8.263. Data for $h = 0$ were taken from [5]. These calculations show that all curves in Figs. 3 and 4 are, within limits of computation accuracy, symmetric about the Y-axis.

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ON THE APPROXIMATIONS AND BIFURCATIONS OF A DYNAMIC SYSTEM

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By means of an example of a classical problem in flight dynamics we examine the influence of approximation on the structure of the partitioning of the phase space and of the parameter space of a dynamic system. For a qualitative investigation of dynamic systems we can use the transition from the original model to a simplified or piece-wise integrable one, by approximating the characteristics in the equations of motion. Here arises the important question of the admissible deviations of the approximating functions from the real characteristics for the preservation of the necessary closeness between the original and the approximating system The concept of necessary closeness is not unique and is determined by the aims of the investigation. For example, it can be understood as the requirement of retaining for the approximating system the same phase space and parameter space partitioning structure as for the original system [1]. In a general formulation the problem reduces to the question of preserving or losing bifurcations during the transition to the approximating system. The difficulties arisfsing here are connected with the fact that not all the bifurcations may be kept track of by regular methods, and furthermore, for "fused"